Topology Vol. 16, pp. 409–415. Pergamon Press, 1977. Printed in Great Britain

COMPLEXES DOMINATED BY A 2-COMPLEX

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Dedicated to the memory of my friend and colleague, George Cooke.

(*Received* 9 *December* 1976)

AN OLD problem concerning *CW* complexes is the following:

PROBLEM. *If X is a CW complex dominated by a finite (respectively n-) complex, is X homotopy equivalent to a finite (respectively n-) complex?*

Wall[10] answered most of this by showing that the answer to finiteness is no in general, with an obstruction lying in $\tilde{K}_0(Z\pi_1X)$. For the dimension question, he showed that the answer is yes if $n > 2$. The Stallings–Swan Theorem[9] answers the problem affirmatively for $n = 1$. We study the case $n = 2$ here. We shall prove in §2

THEOREM 1. *If X is dominated by a 2-complex, then there is a wedge of 2-spheres W such that $X \vee W$ is of the homotopy type of a 2-complex. If X is a finite complex, then W is finite.*

Furthermore, we shall construct (Theorem 4) a 3-complex X and a 2-complex K such that $X \vee S^2 \simeq K \vee S^2$ but $X \not\simeq K$. I suspect, but have not been able to prove, that X is not homotopy equivalent to any 2-complex. This would entirely settle the question being studied.

Along the way we shall prove two other useful results. The first (Theorem 2) is a condition for being able to realize homomorphisms f_i: $\pi_i(K) \to \pi_i(L)$, $i = 1, 2$, where K is a 2-complex and L an arbitrary complex. The second (Theorem 3) gives a relation between stably equivalent modules and stably equivalent complexes.

I wish to thank George Cooke for great help in this study. In particular his Lemma 1 was a motivating force. I also appreciate many very useful conversations with R. G. Swan.

Note. Throughout, we refer to complexes X dominated by a 2-complex. An equivalent condition[10] is that $H^i(X; A) = 0$, $i > 2$ for all π_1X-modules A; i.e., X has cohomological dimension two.

§1. SOME USEFUL RESULTS

Throughout this discussion, complex will mean *CW* complex. X dominated by Y means that there are maps $f: X \to Y$, $g: Y \to X$ with $gf \sim 1_X$. $\sim$ is the symbol for homotopy, $\simeq$ for homotopy equivalence. h will represent the Hurewicz homomorphism. $[X, Y]$ is the set of homotopy classes of maps $X \to Y$.

The following is a generalization of a result of George Cooke for the case $n = 2$.

LEMMA 1. *Let X be an n-complex, $n > 1$, $\pi = \pi_1X$, $\Lambda = Z\pi$. Assume that $\pi_nX \cong M \oplus F$ where F is a free Λ-module with a basis $\{x_\alpha\}$. Then in H_nX, $\{h(x_\alpha)\}$ is the basis of summand if and only if there is a complex Y which is X with $(n+1)$-cells attached such that $\pi_n(Y) \cong M$ and $X \simeq Y \vee \bigvee_\alpha S^n$.*

Proof. The "if" part is immediate. For the converse, let Y be X union $(n+1)$-cells attached by maps which represent the x_α. Then $\pi_nY \cong \pi_n(x)/\Lambda\{x_\alpha\} \cong M$. Now $\operatorname{Hom}(H_nX, Z) \cong H^n(X) \cong [X, S^n]$. Since the $h(x_\alpha)$ generate summands, for some D, $H_n(X) = D \oplus \bigoplus_\alpha (h(x_\alpha))$. Thus for each α there is a homomorphism $H_nX \to Z$ which is 0 on D and on $h(x_\beta)$ for $\beta \neq \alpha$, and sends $h(x_\alpha)$ to 1. Let $f_\alpha: X \to S^n$ realize this homomorphism. Let $i: X \to Y$ be the inclusion.

Define $\varphi_1: X \to Y \times \times_\alpha S^n$ as $(i, (f_\alpha))$. Up to homotopy, we can change φ_1 to φ_2, a cellular

*Research partially supported by the National Science Foundation.

map. Since X is an n-complex, im $\varphi_2(X) \subset n$-skeleton of $Y \times \underset{\alpha}{\times} S^n$ which is $W' = Y \vee \underset{\alpha}{\vee} S^n$ (we assume each S^n has one 0-cell and one n-cell). Let φ': $X \to W'$ be this map. Replace W' by W, the mapping cylinder of W and let φ: $X \to W$ be the inclusion. By construction φ induces a homotopy isomorphism through dimension n. Since X is an n-complex and W an $(n+1)$-complex, this is almost (but not quite) good enough.

Look at the exact sequence

$$\pi_{n+1}X \xrightarrow{\beta} \pi_{n+1}Y \longrightarrow \pi_{n+1}(Y, X) \xrightarrow{\gamma} \pi_n X.$$

$\pi_{n+1}(Y, X)$ is a free Λ-module on generators ω_α where $\gamma(\omega_\alpha) = x_\alpha$. So γ is a monomorphism. Thus β is an epimorphism. Thus $\pi_{n+1}X \to \pi_{n+1}Y$ is also an epimorphism. If $\bar{x}_\alpha$: $S^n \to X$ represents x_α, then $f_\alpha \circ \bar{x}_\alpha \sim$ identity. Thus $\vee\, \bar{x}_\alpha$: $\underset{\alpha}{\vee} S^n \to X$ is such that if it is composed with $X \to W' \to \underset{\alpha}{\vee} S^n$ we get the identity. Thus $X \to \underset{\alpha}{\vee} S^n$ induces a split epimorphism in all dimensions in homotopy. Now $\pi_{n+1}(W') = \pi_{n+1}(Y) \oplus \pi_{n+1}(\underset{\alpha}{\vee} S^n) \oplus$ Whitehead products $[\pi_1(Y), \pi_{n+1}(\underset{\alpha}{\vee} S^n)] \oplus [\pi_2(Y), \pi_n(\underset{\alpha}{\vee} S^n)]$. Since $\pi_{n+1}(\varphi')$ maps onto each piece, it is onto. Thus $\pi_{n+1}(\varphi)$: $\pi_{n+1}(X) \to \pi_{n+1}(W)$ is onto so $\pi_i(W, X) = 0$ for $i \le n+1$. Looking at the universal covers, $\pi_i(\tilde{W}, \tilde{X}) = 0$ for $i \le n+1$ so $H_i(\tilde{W}, \tilde{X}) = 0$ for $i \le n+1$; since $\tilde{X}$ is an n-complex and $\tilde{W}$ an $(n+1)$-complex $H_i(\tilde{W}, \tilde{X}) = 0$ for $i > n+1$. Thus $\pi_i(W, X) = \pi_i(\tilde{W}, \tilde{X}) = H_i(\tilde{W}, \tilde{X}) = 0$ for all i, and φ is a homotopy equivalence, so φ': $X \to Y \vee \underset{\alpha}{\vee} S^n$ is also.

COROLLARY 1. *Let K be a 2-complex, $\pi = \pi_1 K$, $\Lambda = Z\pi$ and $\pi_2 K$ a free Λ-module. If the cohomological dimension of π is 2, then there is an aspherical 3-complex L (i.e., a $K(\pi, 1)$) with 2-skeleton K such that $K \simeq L \vee \vee S^2$.*

Proof. There is an exact sequence (e.g., from the Serre Spectral Sequence of the fibration $\tilde{K} \to K \to K(\pi, 1)$)

$$0 \to H_3\pi \to \pi_2 K \otimes_{Z\pi} Z \xrightarrow{\bar{h}} H_2 K \to H_2\pi \to 0.$$

Since π has cohomological dimension 2, $H_3\pi = 0$ and $H_2\pi$ is free abelian. Thus $\bar{h}$ is an isomorphism onto a summand; i.e., if $\{x_\alpha\}$ is a basis for $\pi_2 K$, then $\{h(x_\alpha)\}$ is a basis for a summand of $H_2 K$. Thus by Lemma 1 we get L with 2-skeleton K, $\pi_2 L = 0$ and $K \simeq L \vee \vee S^2$. Since $\tilde{L}$ is dominated by $\tilde{K}$, a 2-complex, $H_3\tilde{L} = 0$. But $\pi_2\tilde{L} = 0$ and $\tilde{L}$ is a simply-connected 3-complex. Thus $\tilde{L}$ is contractible so L is aspherical and the proof is complete.

Let K be a complex with a single 0-cell. Then the 2-skeleton of K yields a presentation $\{x_i | r_\alpha\}$ of $\pi = \pi_i K$. The CW chains of $\tilde{K}$

$$\cdots \to C_3\tilde{K} \xrightarrow{\partial_3} C_2\tilde{K} \xrightarrow{\partial_2} C_1\tilde{K} \xrightarrow{\partial_1} C_0\tilde{K}$$

may be described as follows: $C_i\tilde{K}$ is a free $\Lambda = Z\pi$-module with basis corresponding to the i-cells of K; for $i = 0$ on one generator e_0, for $i = 1$ on generators b_i and for $i = 2$ on generators c_α. $\partial_1(b_i) = (x_i - 1)e_0$, $\partial_2(c_\alpha) = \Sigma\, d_i(r_\alpha) b_i$ where d_i: $F \to Z\pi$ is the Fox derivative[5] going from the free group F on generators x_i to $Z\pi$ such that $d_i(x_j) = \delta_{ij}$, the Kronecker delta. ($\mathrm{d}(xy) = \mathrm{d}x + x\,\mathrm{d}y$ is satisfied by all derivations.)

Set $\bar{C}_2(\tilde{K}) = C_2(\tilde{K})/\mathrm{im}\, \partial_3$, $\bar{\partial}_2$: $\bar{C}_2\tilde{K} \to C_1\tilde{K}$ induced. Then $\pi_2 K \cong \pi_2\tilde{K} \cong H_2\tilde{K} = \ker \bar{\partial}_2$ so we can think of $\pi_2 K$ as a submodule of $\bar{C}_2\tilde{K}$.

LEMMA 2. *Let K be a 2-complex. Let φ: $C_2\tilde{K} \to C_2\tilde{K}$ be a $Z[\pi_1 K]$-module homomorphism with $\partial_2\varphi = \partial_2$. Let $\bar{\varphi}$: $\pi_2 K \to \pi_2 K$ be the induced map. Then there is a map f: $K \to K$ with $\pi_1 f$ the identity and $\pi_2 f = \bar{\varphi}$.*

Lemma 2 is a special case of Theorem 2 which follows. We have separated it out because the general case requires the following technical definition:

Definition. Let K and L be complexes whose 2-skeleta realize the presentations $\{x_i | u_\alpha\}$ of $\pi = \pi_1 K$ and $\{y_j | v_\beta\}$ of $\rho = \pi_1 L$. Let $\{\alpha_i\}$ and $\{\beta_j\}$ be the bases of $C_1(\tilde{K})$ and $C_1(\tilde{L})$, taken as earlier. Let $f\colon \pi \to \rho$ be a homomorphism. Then $\varphi\colon C_*\tilde{K} \to C_*\tilde{L}$ is *compatible with* f if there are words W_i in the y_i such that

(a) $f(x_i)$ is the image of W_i in ρ,

(b) $\varphi(\alpha_i) = \sum_j (\partial_j W_i)\beta_j$ where $\partial_j y_k = \delta_{jk}$ is the Fox derivative, and

(c) φ is a chain map, equivariant with respect to f.

Then we get the following:

THEOREM 2. *Let K and L be complexes with a single 0-cell, K a 2-complex. Let $f_i\colon \pi_i K \to \pi_i L$, $i = 1, 2$ be homomorphisms. Then there is a map $f\colon K \to L$ inducing the f_i if and only if there is some $\varphi\colon C_*\tilde{K} \to C_*\tilde{L}$ compatible with f_1 such that*

$$\begin{tikzcd}
\pi_2 K \arrow[r] \arrow[d, "f_2"] & C_2\tilde{K} \arrow[d, "\varphi"] \\
\pi_2 L \arrow[r] & C_2\tilde{L}
\end{tikzcd}$$

commutes.

Proof. The "only if" is immediate. For the "if" part assume the hypotheses. Choose the W_i as in the definition. We can assume that for the 1-skeleton $\pi_1 K^1$ and $\pi_1 L^1$ are free on $\{x_i\}$ and $\{y_i\}$. Let $\hat{g}\colon \pi_1 K^1 \to \pi_1 L^1$ be given by $\hat{g}(x_i) = W_i(y_j)$. We have the exact diagram

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d, "q"] & \pi_1 K^1 \arrow[r, "p_K"] \arrow[d, "\hat{g}"] & \pi_1 K \arrow[d, "f_1"] \\
1 \arrow[r] & M \arrow[r] & \pi_1 L^1 \arrow[r, "p_L"] & \pi_1 L
\end{tikzcd}$$

where N and M are the kernels of p_K and p_L.

We also have the diagram

(*)
$$\begin{tikzcd}
0 \arrow[r] & \pi_2 K \arrow[r] \arrow[d, "f_2"] & \pi_2(K, K^1) \arrow[r] \arrow[d, dashed, "k"] & \pi_1 K^1 \arrow[d, "\hat{g}"] \\
0 \arrow[r] & \pi_2 L \arrow[r] & \pi_2(L, L^1) \arrow[r] & \pi_1(L^1)
\end{tikzcd}$$

Assume for now that there exists a k making this diagram commute. Then we can define $f\colon K \to L$ as follows: because K^1, L^1 are wedges of circles $\hat{g}$ may be realized as $g\colon K^1 \to L^1$. Let e be any 2-cell of K and let $\xi\colon (D^2, \dot{D}^2) \to (K, K^1)$ be its characteristic map. $[\xi] \in \pi_2(K, K^1)$ so we can find $l\colon (D^2, \dot{D}^2) \to (L, L^1)$ representing $k[\xi]$. $g \circ \xi|\dot{D}^2 \sim l|\dot{D}^2$ so by the homotopy extension theorem we may change l up to homotopy to l' where $g \circ \xi|\dot{D}^2 = l'|\dot{D}^2$. Now define $f|\mathring{e}$ by $l' \circ \xi^{-1}$. This matches up with g yielding $f\colon K \to L$ inducing f_1 and f_2.

We now need to construct k. We need to know the following facts[11].

(1) $\pi_2 K$ lies in the center of $\pi_2(K, K^1)$.

(2) If K yields the presentation $\{x_i | r_\alpha\}$ of $\pi = \pi_1 K$, then $\pi_2(K, K^1)$ is the group generated by xs_α, $x \in \pi_1 K^1$, s_α a symbol, with the relations $xs_\alpha + ys_\beta - xs_\alpha = (xr_\alpha x^{-1}y)s_\beta$; that is, $\pi_2(K, K^1)$ is the free crossed module on the s_α over $\pi_2(K, K^1) \to \pi_1(K^1)$.

(3) In particular, its abelianization, $\pi_2(K, K^1)^a \cong C_2\tilde{K}$.

Thus $\varphi\colon C_2\tilde{K} \to C_2\tilde{L}$ can be considered as a map $\pi_2(K, K^1)^a \to \pi_2(L, L^1)^a$.

Thus we have the commutative diagram

(**)
$$\begin{tikzcd}
\pi_2(K, K^1) \arrow[rrr, "\partial_K"] \arrow[dr] \arrow[ddd, dashed, "k"'] & & & N \arrow[dl] \arrow[ddd, "q"] \\
& \pi_2(K, K^1)^a \arrow[r] \arrow[d, "\varphi"] & N^a \arrow[d, "q^a"] & \\
& \pi_2(L, L^1)^a \arrow[r, "\partial_L{}^a"] & M^a & \\
\pi_2(L, L^1) \arrow[rrr, "\partial_L"] \arrow[ur] & & & M \arrow[ul]
\end{tikzcd}$$

Let us construct k making this diagram commute: since $\pi_2(K, K^1)$ is a free crossed module and φ and q^a are f_1-equivariant we need only describe $k(s_\alpha)$. Let $n \in \pi_2(L, L^1)^a$ be $\varphi(\bar{s}_\alpha)$ and $m \in M$ be $q\partial_K(s_\alpha)$. Since they both map to the same thing in M^a and ∂_L is onto, there is some $t_\alpha \in \pi_2(L, L^1)$ which maps to n and m. Let $k(s_\alpha) = t_\alpha$. This defines k.

We have constructed k so that the diagram (**) commutes. But

$$\begin{tikzcd} \pi_2 K \arrow[r] \arrow[d, "f_2"] & \pi_2(K, K^1)^a \arrow[d, "\varphi"] \\ \pi_2 L \arrow[r] & \pi_2(L, L^1)^a \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \pi_2(K, K^1) \arrow[rr] \arrow[dr] & & \pi_1 K^1 \arrow[ddd, "\hat{g}"] \\ & N \arrow[ur] \arrow[d, "q"] & \\ & M \arrow[dr] & \\ \pi_2(L, L^1) \arrow[ur] \arrow[rr] & & \pi_1 L^1 \end{tikzcd}$$

commute so diagram (*) commutes and we are done.

§2. PROOF OF THEOREM 1

If X is dominated by a 2-complex, it is a fortiori dominated by a 3-complex, hence by Wall's result[10] is of the homotopy type of a 3-complex. Then Theorem 1 follows from the following corollary to Lemma 1:

COROLLARY 2. *If X is an $(n+1)$-complex dominated by an n-complex, then there is a wedge of n-spheres W such that $X \vee W \simeq X^n$, the n-skeleton of X. Furthermore $SW \simeq X/X^n$.*

Proof. Assume $f'\colon X \to Y$ and $g'\colon Y \to X$ are given so that $g' \circ f' \sim 1_X$ and Y is an n-complex. $g' \sim g''$ a cellular map. Since g'' is simplicial $g''(Y) \subset X^n$. Let $g\colon Y \to X^n$ be given by $g'' \circ i$, $i\colon X^n \hookrightarrow X$ the inclusion. Let $f = g \circ f'$. Then $i \circ f = i \circ g \circ f' = g'' \circ f' \sim g' \circ f' \sim 1_X$. Thus X homotopy retracts to X^n so $\pi_n(X^n) \cong \pi_n(X) \oplus \pi_{n+1}(X, X^n)$ and $H_n(X^n) \cong H_n(X) \oplus H_{n+1}(X, X^n)$ where both splittings are induced naturally from the maps i and f, hence they respect the Hurewicz map. $\pi_{n+1}(X, X^n)$ (resp. $H_{n+1}(X, X^n)$) are free $Z\pi_1 X$- (resp. Z-) modules on the cells of $X - X^n$. The Hurewicz map is a bijection of bases and X is X^n with $(n+1)$-cells attached to kill $\pi_{n+1}(X, X^n)$, so by Lemma 1, $X \vee W \simeq X^n$ where W is a wedge of n-spheres indexed by the cells of $X - X^n$, whence $SW \simeq X/X^n$.

Note. Instead of using Lemma 1, here is an outline of an alternative direction using Theorem 1.1 of [4]:

THEOREM (M. Dyer). *A connected 3-complex Y has the homotopy type of a connected 2-complex if and only if $H_3(\tilde{Y}) = 0$ and there is a connected 2-complex W such that $\pi_i(Y) \cong \pi_i(W)$, $i = 1, 2$ and the k-invariant in $H^3(\pi_1(Y), \pi_2(Y)) \cong H^3(\pi_1(W); \pi_2(W))$ are the same.*

Now if X is a 3-complex dominated by a 2-complex, then following the ideas as above one can show that $X \vee W$ and X^2 have not only the same π_1 and π_2, but the same k-invariant. Thus by Dyer's theorem $X \vee W \simeq$ 2-complex and thus $X \vee W \simeq X^2$.

§3. REALIZING STABLY EQUIVALENT MODULES

Λ-modules M and N are said to be stably equivalent, $M \simeq N$, if $M \oplus F \cong N \oplus F$ for some finitely-generated free module F. (I do not know of any case where rank $F > 2$ is necessary.)

THEOREM 3. *Let K be a 2-complex with $\pi_1 K = \pi$, a finite group. Let N be a $Z\pi$-module such that $\pi_2 K \simeq N$. Then there is a 3-complex X such that $X \vee W$ where W is a finite wedge of 2-spheres and $\pi_2 X \cong N$.*

Remarks. (1) W may be taken as a wedge of k 2-spheres where $k = \operatorname{rank} F$ where $\pi_2 K \oplus F \cong N \oplus F$.

(2) In the proof X is taken as the cone of an obviously defined map $W \to K \vee W$. If π is infinite the similarly defined X with $\pi_2 X \cong N$ will *not* necessarily have the property that $X \vee W \simeq K \vee W$ although it is possible that another X with $\pi_2 X \cong N$ may be found with $X \vee W \simeq K \vee W$. A counterexample is the following: Let K be the 2-complex presented by

$\{x, y, z|[x, y], [y, z], [z, x]\}$. K is the 2-skeleton of the 3-torus. A straightforward calculation shows that $\pi_2 K \cong \Lambda = Z\pi_1 K$. Let $N = \Lambda$. Of course $N \oplus \Lambda \cong \pi_2 K \oplus \Lambda$ but take the isomorphism to be $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the twist. Then the X that would be defined is of the form $X = (K \cup e^3) \vee S^2$. Since $H^*(X; Z) \not\cong H^*(K; Z)$ as rings, $X \vee S^2 \not\simeq K \vee S^2$. Of course in this case $X = K$ will work, but the point is that for π finite, any X so constructed will work.

Proof of Theorem 3. Let $M = \pi_2 \tilde{K}$. Let $C_* = C_* \tilde{K}$, $\partial = \partial_2$, $I = \operatorname{im} \partial_2$. Then $0 \to M \xrightarrow{f} C_2 \xrightarrow{\partial} I \to 0$ is an exact sequence. Now $\pi_2(K \vee W) = M \oplus F \cong N \oplus F$. Let $\{(m_i, x_i) \in M \oplus F\}_i$ be a basis for F in the splitting $N \oplus F$. Let $h: W \to K \vee W$ be a map which represents $(m_i, x_i) \in \pi_2(K \vee W)$ on the ith sphere. Let X be the mapping cone of h—i.e., X is $K \vee W$ with 3-attached to kill the (m_i, x_i). Thus $\pi_2 X \cong N$.

Let $\bar{\alpha}: F \to M$, $\beta: F \to F$ be given by $\bar{\alpha}(b_i) = m_i$, $\beta(b_i) = x_i$, and let $\alpha: F \to C_2$ be $f \circ \bar{\alpha}$. Look at the following exact diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & F & = & F & & \\
 & & \downarrow{\scriptstyle (\bar{\alpha}, \beta)} & & \downarrow{\scriptstyle (\alpha, \beta)} & & \\
0 \to & & M \oplus F & \to & C_2 \oplus F & \to I \to & 0 \\
 & & \downarrow & & \downarrow & \| & \\
0 \to & & N & \to & Cok(\alpha, \beta) & \to I \to & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

N and I are free abelian groups, because $N \subset M \oplus F \subset C_2 \oplus F$ and $I \subset C_1$. Thus $Cok(\alpha, \beta)$ is a free abelian group so $F \xrightarrow{(\alpha,\beta)} C_2 \oplus F$ splits over Z. But for a finite group ring free (or projective) implies weakly injective (cf. [1])—i.e., monomorphisms which split over Z split over Λ. Thus there exist maps $\mu: C_2 \to F$, $\lambda: F \to F$ such that $\mu\alpha + \lambda\beta = 1_F$.

Look at the following endomorphism of $C_*(K \vee W \vee W)$:

$$\begin{array}{ccccc}
C_2 \oplus F \oplus F & \xrightarrow{(\partial, 0, 0)} & C_1 & \longrightarrow & C_0 \\
\begin{pmatrix} 1 & \alpha & 0 \\ 0 & \beta & 1 \\ \mu & 0 & \lambda \end{pmatrix} \Big\downarrow & & \| & & \| \\
C_2 \oplus F \oplus F & \xrightarrow[(\partial, 0, 0)]{} & C_1 & \longrightarrow & C_0.
\end{array}$$

This is an automorphism since

$$\begin{pmatrix} 1 & \alpha & 0 \\ 0 & \beta & 1 \\ \mu & 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 - \alpha\mu & -\alpha\lambda & \alpha \\ \mu & \lambda & -1 \\ -\beta\mu & 1 - \beta\lambda & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By Lemma 2, this can be realized by a homotopy equivalence $k: K \vee W \vee W \to K \vee W \vee W$. Observe that $\pi_2(k)$ is the automorphism of $\pi_2(K \vee W \vee W) = M \oplus F \oplus F$ given by the matrix $\begin{pmatrix} 1 & \bar{\alpha} & 0 \\ 0 & \beta & 1 \\ \mu \circ f & 0 & \lambda \end{pmatrix}$. Thus if k is composed with the map $i: W \to K \vee W \vee W$, inclusion into the second factor, we get $j \circ h$, where $j: (K \vee W) \to (K \vee W) \vee W$ is inclusion into the first factor. So $j \circ h = k \circ i$. Since k is a homotopy equivalence, the mapping cones $C_{j \circ h}$ and C_i are homotopy equivalent. But $C_i = K \vee D \vee W$ where D is a wedge of free 2-discs. Thus $C_i \simeq K \vee W$. $C_{j \circ h} = C_h \vee W = X \vee W$. Thus $X \vee W \simeq K \vee W$.

§4. CONSTRUCTING THE EXAMPLE

Throughout this section let $\pi = \langle x, y | yxy^{-1}x, y^2x^8 \rangle$, the generalized quaternionic group of order 32. Let $\Lambda = Z\pi$. We shall prove:

PROPOSITION 1. *There is a Λ-module M which is not cyclic, but $M \oplus \Lambda \cong \Lambda/Z \oplus \Lambda$.*

Remark. Z is the trivial submodule of Λ generated by $\varphi = \Sigma_{\alpha \in \pi} \alpha$.

Proof of Proposition 1. In $\mathbf{C}$, the complex numbers, let $\zeta = e^{2\pi i/16}$ and $\tau = \zeta + \zeta^{-1} = \sqrt{2+\sqrt{2}}$. Let $\tau' = \sqrt{2-\sqrt{2}}$, a conjugate of τ. $\tau = 2 \cos \pi/8$, $\tau' = 2 \sin \pi/8$, $\tau\tau' = \sqrt{2}$ and $2\zeta = \tau + i\tau'$. Let P be the subring of the quaternions generated by τ, ζ, and $\beta = \tau'^{-1}(1+j)$. P is an algebra over $Z[\tau]$. (Note that $(\tau^2-2)^2 = 2$.) As a $Z[\tau]$-module P is generated by 1, ζ, β, and $\zeta\beta$. Since $(\zeta\tau - 1)^2 = i$, $\tau' = i(\tau - 2\zeta) \in P$ so $j = \tau'\beta - 1 \in P$. Thus there is a ring homomorphism $\xi: \Lambda \to P$ with $\xi(x) = \zeta$ and $\xi(y) = j$, well-defined since $\zeta^8 = -1 = j^2$ and $j\zeta j^{-1} = \zeta^{-1}$. Since $\xi(x^8+1) = 0$ and $\varphi = (1 + x + \cdots + x^7)(x^8+1)(y+1)$ we have $\xi(\varphi) = 0$.

Swan[8] constructed a projective ideal $I \subset \Lambda$ and an exact sequence $0 \to \Lambda \xrightarrow{\theta} \Lambda \oplus \Lambda \to I \to 0$ (whence $I \oplus \Lambda \cong \Lambda \oplus \Lambda$) and proved that $P \otimes_\Lambda I \not\cong P$. So that in particular $I \not\cong \Lambda$. Now the fixed points I^π of I must be isomorphic to Z because $I^\pi \oplus Z = I^\pi \oplus \Lambda^\pi = (I \oplus \Lambda)^\pi \cong (\Lambda \oplus \Lambda)^\pi = Z \oplus Z$. Let $M = I/I^\pi$. It is immediate that $M \oplus \Lambda \cong \Lambda/Z \oplus \Lambda$. On the other hand since $\xi(\varphi) = 0$, $P \otimes_\Lambda M \cong P \otimes_\Lambda I \not\cong P \cong P \otimes_\Lambda (\Lambda/Z)$ so $M \not\cong \Lambda/Z$.

Now let K be the 2-complex corresponding to the presentation $\{x, y | yxyx^{-1}, y^2x^8\}$ of π.

PROPOSITION 2. $\pi_2 K \cong \Lambda/Z$.

Proof. If we form ∂_2 as earlier we see that it has the matrix form $\begin{pmatrix} 1+xy & \sigma \\ x-1 & y^2+y^3 \end{pmatrix}$ where $\sigma = 1 + x + x^2 + \cdots + x^7$. By considering Λ as a Γ-module where $\Gamma = Z\langle x | x^{16}\rangle$ we can manipulate the equations and we find that $\ker \partial_2$ is generated by $(y - x^7, (x^8 - x^7)y)$ and $(1 - x^8, (x-1)(1-y))$. Operating on $\begin{pmatrix} y - x^7 & 1 - x^8 \\ (x^8 - x^7)y & (x-1)(1-y) \end{pmatrix}$ by row and column operations we get $\begin{pmatrix} 1-y & 0 \\ 1-x & 0 \end{pmatrix}$.

Thus $\pi_2 K \cong \ker \partial_2$ is the cyclic submodule of $\Lambda \oplus \Lambda$ generated by $\gamma = (1-y, 1-x)$. If $\lambda \in \Lambda$, $\lambda\gamma = 0$ if and only if $\lambda = \lambda y$ and $\lambda = \lambda x$. Thus $\lambda\gamma = 0$ if and only if γ is in the trivial subgroup Z of Λ. Thus $\ker \partial_2 \cong \Lambda/Z$.

Remark. Another way of seeing this is to recall[7] that π acts freely (and orthogonally) on S^3. So K is the 2-skeleton of a 3-manifold $M = K \cup e^3$. Since $\pi_2 M = \pi_2 S^3 = 0$, $\pi_2 K$ must be cyclic. $0 \to \pi_2 K \to \Lambda^2 \to \Lambda^2 \to \Lambda \to Z \to 0$ is exact. Tensor this sequence with $\mathbf{Q}$. Since $\mathbf{Q}\pi$ is semi-simple[1], the sequence splits so that $\mathbf{Q} \otimes \pi_2 K \cong \mathbf{Q} \otimes (\Lambda/Z)$. Since $\pi_2 K$ is cyclic, it is Λ/J for some ideal J. Since $\Lambda/J \subset \mathbf{Q} \otimes \Lambda/J \cong \mathbf{Q} \otimes \Lambda/Z$ which is fixed-point free, $Z \subset J$ so $\Lambda/Z \to \Lambda/J$ is an epimorphism of free abelian groups of the same finite rank, hence it is an isomorphism.

Now putting Propositions 1 and 2 together, Theorem 3 gives us the example:

THEOREM 4. *There is a 3-complex X with $\pi_2 X \cong M$ and $X \vee S^2 \simeq K \vee S^2$ where K is a 2-complex with $\pi_2 K \cong \Lambda/Z \not\cong M$.*

Note. There is a similar example due to Dunwoody[3], but over an infinite group. In that case his X turns out to be homotopy equivalent to a 2-complex, but he uses very strongly the fact that for an infinite group one can get totally different presentations of the same group. More precisely: if F is a finitely generated free group and $N \lhd F$ let a_N be the minimum number of generators required for the $Z[F/N]$-module N^a. Gruenberg shows [6] that if $F/N \cong F/K$ is finite then $a_N = a_K$, but Dunwoody's example comes from an infinite group $F/N \cong F/K$ where $a_N \neq a_K$.

A reason for believing that $X \not\simeq$ a 2-complex is that the module M is "bad" because the following fails:

Definition. Let σ be a finite group of order $s \neq 1$. A module N is a Swan module if $\mu(N) = \max_{\text{prime } p | s} \mu(Z_p \otimes N)$, where $\mu(N)$ is the minimum number of generators.

If L is a 2-complex $\pi_2 L$ should be a very restricted and "nice" class of modules. One expects them to be Swan modules. The module M, above, is not: $Z_p \otimes M \cong Z_p \otimes (\Lambda/Z)$ for all primes p, so $\mu(Z_p \otimes M) = 1$, but $\mu(M) = 2$. A module closely related to $\pi_2 L$ is R^a where $\pi_1 L = F/R$ a finitely generated free group F modulo a normal subgroup R. R^a, the abelianization, is always a Swan module [2, 6] for $\pi_1 L$ finite.

I suspect that the only exception to the general rule that X dominated by a 2-complex implies that X is a 2-complex will be almost exactly as in the (supposed) example presented here: Let π be a finite group that has a periodic free resolution of period 4 (equivalently π acts freely on a homotopy S^3). Then (and only then) there is a 2-complex K such that $\pi_1 K \cong \pi$ and $\pi_2 K \cong Z\pi/Z$. Corresponding to a non-cyclic module $M \doteq Z\pi/Z$ there exists a complex X which is dominated by a 2-complex, but I suspect is not $\simeq$ a 2-complex.

I am guessing that if X is dominated by a 2-complex and $\pi_1 X$ is finite, then the obstruction to X being $\simeq$ a 2-complex is whether or not $\pi_2 X$ requires the "right" number of generators. By a result in [2], $M \doteq N$ implies that they both need the same number of generators except in the case that one is cyclic and the other non-cyclic.

For $\pi_1 X$ infinite, Dunwoody's example suggests that there may be enough "room" to build a 2-complex X with $\pi_2 X \cong M$ whenever $M \doteq \pi_2 K$ for some 2-complex K.

Note added in proof: I have learned that J. G. Ratcliffe (M.I.T.) has an independent proof of Theorem 1 in his 1977 thesis (U. of Michigan). His method is apparently different and proceeds from a study of free and projective crossed modules.

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